Supplementary: Surface-Only Ferrofluids

LIBO HUANG, KAUST
DOMINIK L. MICHELS, KAUST

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S1 GALERKIN BEM

This section serves as an introduction to the Galerkin boundary element method. We refer, e.g., to Rjasanow and Steinbach [2007] for a more rigorous introduction. We consider the boundary value problem of the Laplace equation.

$$\nabla^2 u(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \Omega \in \mathbb{R}^3 . \tag{S1}$$

The boundary element method relies on the representation formula of the solution to the Laplace equation (see Rjasanow and Steinbach [2007], Eq. (1.6)):

$$u(\boldsymbol{x}) = \int_{\Gamma=\partial\Omega} G(\boldsymbol{x},\boldsymbol{y}) \frac{\partial u(\boldsymbol{y})}{\partial n} ds_y - \int_{\Gamma} \frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{y})} u(\boldsymbol{y}) ds_y \,, \boldsymbol{x} \in \Omega \,, \tag{S2}$$

$$G(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi} \frac{1}{\|\boldsymbol{x}-\boldsymbol{y}\|} , \quad \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^3 , \tag{S3}$$

$$\frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{y})} = \frac{1}{4\pi} \frac{\boldsymbol{n}(\boldsymbol{y}) \cdot (\boldsymbol{x}-\boldsymbol{y})}{\|\boldsymbol{x}-\boldsymbol{y}\|^3} , \tag{S4}$$

where the Green's function G denotes the fundamental solution of the Laplace equation in three dimensions and $\partial/\partial n = \boldsymbol{n} \cdot \nabla$ is the directional derivative on the boundary normal pointing outwards.

There are more general forms of the two terms in Eq. (S2):

$$(\widetilde{V} w)(\boldsymbol{x}) = \int_{\Gamma} G(\boldsymbol{x},\boldsymbol{y}) w(\boldsymbol{y}) ds_y , \quad \boldsymbol{x} \in \Omega , \tag{S5}$$

$$(W v)(\boldsymbol{x}) = \int_{\Gamma} \frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{y})} v(\boldsymbol{y}) ds_y , \quad \boldsymbol{x} \in \Omega . \tag{S6}$$

The quantities $(\widetilde{V} w) \in \mathbb{R}$ and $(W v) \in \mathbb{R}$ are defined for a point $\boldsymbol{x}$ inside the domain Ω. They are called single layer respectively double layer potential because they resemble the electric field generated by a single layer of electric charges on the surface Γ and two layers of opposite charges on the surface. They transform the single layer charges $w(\boldsymbol{y}) \in \mathbb{R}$ and double layer charges $v(\boldsymbol{y}) \in \mathbb{R}$ defined on the surface Γ to potentials defined inside the domain Ω.

Now returning to the representation formula of the solution (see Eq. (S2)), it is natural to define the boundary value u and the normal derivative $\partial u/\partial n$ as the interior limit at the boundary of corresponding variables defined inside the domain. To set up equations dealing with the values on the boundary, we need to find the interior limit of the values and normal derivatives of the single and double layer potentials. The two limits are related to the interior trace operator γ_0^{int} and interior co-normal derivative γ_1^{int} (see Steinbach [2007], Eq. (1.3) (1.7)):

$$\gamma_0^{\text{int}} f(\boldsymbol{x}) = \lim_{\tilde{\boldsymbol{x}} \in \Omega \to \boldsymbol{x} \in \Gamma} f(\tilde{\boldsymbol{x}}) , \tag{S7}$$

$$\gamma_1^{\text{int}} f(\boldsymbol{x}) = \lim_{\tilde{\boldsymbol{x}} \in \Omega \to \boldsymbol{x} \in \Gamma} \boldsymbol{n}(\boldsymbol{x}) \cdot \nabla f(\tilde{\boldsymbol{x}}) . \tag{S8}$$

We could use the trace operator and co-normal derivative to re-write the representation formula (see Eq. (S2)):

$$u(\boldsymbol{x}) = \widetilde{V}(\gamma_1^{\text{int}} u)(\boldsymbol{x}) - W(\gamma_0^{\text{int}} u)(\boldsymbol{x}), \boldsymbol{x} \in \Omega . \tag{S9}$$

If we apply the trace operator and co-normal derivative to both sides of the equation, we could in the end obtain two integral equations

$$\gamma_0^{\text{int}} u(\boldsymbol{x}) = \gamma_0^{\text{int}} \widetilde{V}(\gamma_1^{\text{int}} u)(\boldsymbol{x}) - \gamma_0^{\text{int}} W(\gamma_0^{\text{int}} u)(\boldsymbol{x}), \boldsymbol{x} \in \Gamma , \tag{S10}$$

$$\gamma_1^{\text{int}} u(\boldsymbol{x}) = \gamma_1^{\text{int}} \widetilde{V}(\gamma_1^{\text{int}} u)(\boldsymbol{x}) - \gamma_1^{\text{int}} W(\gamma_0^{\text{int}} u)(\boldsymbol{x}), \boldsymbol{x} \in \Gamma . \tag{S11}$$

These two equations are called boundary integral equations (BIE).

Assume that $\boldsymbol{x} \in \Gamma$ is locally flat, we can calculate the interior value limit (interior trace) and interior normal derivative limit (interior co-normal derivative) at $\boldsymbol{x}$ for both, single layer potential and double layer potential, as follows:

$$\gamma_0^{\text{int}}(\widetilde{V} w)(\boldsymbol{x}) = (V w)(\boldsymbol{x}) , \tag{S12}$$

$$\gamma_1^{\text{int}}(\widetilde{V} w)(\boldsymbol{x}) = \frac{1}{2} w(\boldsymbol{x}) + (K' w)(\boldsymbol{x}) , \tag{S13}$$

$$\gamma_0^{\text{int}}(W v)(\boldsymbol{x}) = -\frac{1}{2} v(\boldsymbol{x}) + (K v)(\boldsymbol{x}) , \tag{S14}$$

$$\gamma_1^{\text{int}}(W v)(\boldsymbol{x}) = -(D v)(\boldsymbol{x}) . \tag{S15}$$

Please note, that for the four equations above, the functions are all defined on Γ. On the right hand side, we introduce a set of integral operators since they are actually what we can use for the calculation. Their explicit forms are as follows:

$$(V w)(\boldsymbol{x}) = \frac{1}{4\pi} \int_{\Gamma} \frac{w(\boldsymbol{y})}{\|\boldsymbol{x}-\boldsymbol{y}\|} ds_y , \tag{S16}$$

$$(K' w)(\boldsymbol{x}) = \frac{1}{4\pi} \lim_{\varepsilon \to 0} \int_{\boldsymbol{y} \in \Gamma : \|\boldsymbol{y}-\boldsymbol{x}\| > \varepsilon} \frac{(\boldsymbol{y}-\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x})}{\|\boldsymbol{x}-\boldsymbol{y}\|^3} w(\boldsymbol{y}) ds_y , \tag{S17}$$

$$(K v)(\boldsymbol{x}) = \frac{1}{4\pi} \lim_{\varepsilon \to 0} \int_{\boldsymbol{y} \in \Gamma : \|\boldsymbol{x}-\boldsymbol{y}\| > \varepsilon} \frac{(\boldsymbol{x}-\boldsymbol{y}) \cdot \boldsymbol{n}(\boldsymbol{y})}{\|\boldsymbol{x}-\boldsymbol{y}\|^3} v(\boldsymbol{y}) ds_y . \tag{S18}$$

In the equations above, V is called the single layer operator. It is the potential taking single layer charges on the boundary Γ into account. K is called the double layer operator. It is the potential induced by double layer charges outside the infinitesimal neighborhood near $\boldsymbol{x}$. Please note, that its integral domain has an increasingly smaller hole near $\boldsymbol{x}$. In such a hollow integral domain the kernel is not singular. The local contribution of double layer charges v in

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Authors' addresses: Libo Huang, KAUST, Visual Computing Center, Thuwal 23955, KSA; Dominik L. Michels, KAUST, Visual Computing Center, Thuwal 23955, KSA.

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the infinitesimal neighborhood is included in the $-v/2$ term. K' is called the adjoint double layer operator. As the adjoint operator of K, it accounts for the normal derivatives induced by single layer charges except those near $\boldsymbol{x}$. Similarly, the local contribution of the normal derivative induced by single layer charges w is included in the $w/2$ term.

The hypersingular operator D is related to the normal derivatives of the double layer potential. It does not even have an explicit representation as a Cauchy singular surface integral. However, we can evaluate it in the weak sense. Consider two continuous functions u and v defined on Γ. We obtain the following relationship:

$$\begin{aligned}\langle u, Dv\rangle_\Gamma &= \int_\Gamma\int_\Gamma u(\boldsymbol{x})Dv(\boldsymbol{y})ds_y ds_x\,,\\ &= \frac{1}{4\pi}\int_\Gamma\int_\Gamma \frac{\mathrm{curl}_\Gamma u(\boldsymbol{y})\cdot\mathrm{curl}_\Gamma v(\boldsymbol{x})}{\|\boldsymbol{x}-\boldsymbol{y}\|}ds_y ds_x\,, \quad \text{(S19)}\end{aligned}$$

where

$$\mathrm{curl}_\Gamma u(\boldsymbol{x}) = \boldsymbol{n}(\boldsymbol{x})\times\nabla_{\boldsymbol{x}}\tilde{u}(\boldsymbol{x}) \quad \text{(S20)}$$

is the surface curl operator and $\tilde{u}$ defined in $\mathbb{R}^3$ is an arbitrary extension of u defined on Γ.

Ideally, after we choose one of the boundary integral equations, it holds everywhere on the boundary. To solve it numerically, we could only find the best solution in a finite-dimensional space. In our cases, we use triangles to represent the boundary Γ. In order to get a smooth solution, the single layer charges w are at least piece-wise constant on each triangle, while the double layer charges must be continuous on the boundary Γ.

Suppose the mesh consists of N_f faces and N_v vertices, we can define N_f piece-wise constant boundary elements ψ^0 (Figure S1 right) on each face, and N_v piece-wise linear boundary elements ψ^1 (Figure S1 left) for each vertex. Each piece-wise linear boundary element associated with a vertex can can be further decomposed resulting in linear functions defined on its adjacent triangles.

The solution consists of single layer charges w and double layer charges v approximated by these boundary elements:

$$w = \sum_{i=1}^{N_f} w_i\psi_i^0\,,\ v = \sum_{i=1}^{N_v} v_i\psi_i^1\,. \quad \text{(S21)}$$

To find the values of coefficients w_i and v_i, there are two common approaches. They are called collocation boundary element method and Galerkin boundary element method respectively. The Galerkin boundary element method is more costly, but also more accurate, and its convergence and consistency is well studied. For the collocation boundary element method, the boundary integral equations are expected to hold exactly at several points on the surface, for example on the vertices.[1] For the Galerkin boundary element method, the boundary integral equation holds in a weak sense. Both sides of the continuous boundary integral equations are multiplied by a test function and then integrated over the whole domain. This integral is expected to be the same for both sides. The test function is the same as the basis function ψ^0 respectively ψ^1. Using a series of different test functions, we obtain a series of linear equations about the coefficients w_i and v_i. Solving this set of linear equations gives us the best single and double layer charge coefficients w_i and v_i.

[1]Please note, that interior solid angles have to be taken into account.

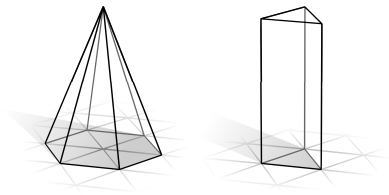

Fig. S1. Continuous piece-wise linear basis ψ^1 defined on vertices (left) and discontinuous piece-wise constant basis ψ^0 defined on triangles (right). The superscript (1 and 0) denotes the maximal order of the polynomials. The grid on the floor indicates the triangle meshes, the vertical direction indicates the strength of the basis function.

As a concrete example, we consider a Neumann problem. Given the normal derivatives $\gamma_1^{\mathrm{int}}u = \sum_{i=1}^{N_f} g_i\psi_i^0$ on the boundary, we look for unknown potentials $\gamma_0^{\mathrm{int}}u$ on the boundary. We use the boundary integral equation derived from the co-normal derivative of the potential (see Eq. (S11)):

$$\gamma_1^{\mathrm{int}}u(\boldsymbol{x}) = \gamma_1^{\mathrm{int}}\widetilde{V}(\gamma_1^{\mathrm{int}}u)(\boldsymbol{x}) - \gamma_1^{\mathrm{int}}W(\gamma_0^{\mathrm{int}}u)(\boldsymbol{x}), \boldsymbol{x}\in\Gamma\,. \quad \text{(S22)}$$

Using the boundary conditions one obtains

$$\sum_{i=1}^{N_f} g_i\psi_i^0(\boldsymbol{x}) = \sum_{i=1}^{N_f}\left(\frac{1}{2}g_i\psi_i^0 + g_iK'\psi_i^0\right)(\boldsymbol{x}) + \sum_{j=1}^{N_v}(Dv_j\psi_j^1)(\boldsymbol{x})\,. \quad \text{(S23)}$$

Then, we multiply both sides of the equation by a test function ψ_k^1 and integrate over the whole surface Γ:

$$\int_\Gamma \psi_k^1(\boldsymbol{x})\left(\sum_{i=1}^{N_f}\left(\frac{1}{2}g_i\psi_i^0 - g_iK'\psi_i^0\right)(\boldsymbol{x}) - \sum_{j=1}^{N_v}(Dv_j\psi_j^1)(\boldsymbol{x})\right)ds_x = 0\,, \quad \text{(S24)}$$

respectively, in a more compact notation,

$$\sum_{j=1}^{N_v}\langle\psi_k^1, D\psi_j^1\rangle_\Gamma v_j = \sum_{i=1}^{N_f}\left(\frac{1}{2}\langle\psi_k^1,\psi_i^0\rangle_\Gamma - \langle\psi_k^1, K'\psi_i^0\rangle_\Gamma\right)g_i\,. \quad \text{(S25)}$$

For N_v test functions ψ_k^1, we can write N_v linear equations. The inner products such as $\langle\psi_k^1, D\psi_j^1\rangle_\Gamma$ represent matrices. To evaluate the value of these entries, it involves an inner integral over the support of the basis function and an outer integral over the support of the test function. Usually, pure numerical quadrature rules [Sauter and Schwab 2010], or the combination of numeric quadrature rules and analytical solutions [Rjasanow and Steinbach 2007; Steinbach 2007], are combined to calculate the value of these entries. By solving this set of linear equations, we can obtain the Dirichlet solution up to a constant difference. The constant is in the null space of the hypersingular operator D as well as the double layer operator K.

S2 ANALYTICAL INTEGRATION OF GREEN'S FUNCTIONS OF LAPLACE EQUATION

In this section, we compile the necessary formulas of analytical integration of the Green's function and its gradient over triangles with constant or linearly varying sources to help the reader to

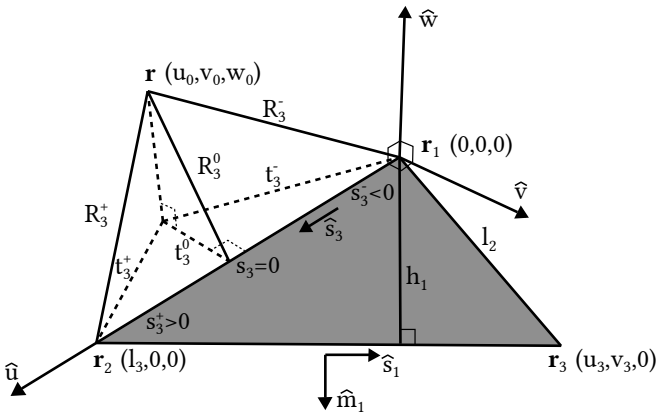

Fig. S2. Definition of variables used in the analytical integration.

reproduce our approach. By default, the formulas below are all adapted from Eibert and Hansen [1995] and Graglia [1993].

S2.1 Definition of Variables

Figure S2 illustrates the definition of variables used in the analytical integration. Assume there is a signed triangle whose vertices are $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3$. This triangle will be the domain over which the Green's function and its gradient are integrated. We evaluate the integral at a field point $\boldsymbol{r}$.

All the computations are carried out in the local frame $\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}$, with origin at the first vertex $\boldsymbol{r}_1$. In this local coordinate system, $\hat{\boldsymbol{w}}$ is effectively the direction of the outward normal $\boldsymbol{n}$ of this triangle. Opposed to each vertex $\boldsymbol{r}_i$ is the i-th edge whose length is l_i, with corresponding height h_i, and outward edge normal direction $\hat{\boldsymbol{m}}_i$ in the triangle plane. The i-th edge of the triangle is associated with a positive direction $\hat{\boldsymbol{s}}_i$ (direction is counter-clockwise if looking from the outward normal direction of the triangle).

In this coordinate system, the field point $\boldsymbol{r}$ at which the Green's function is evaluated has the coordinates (u_0, v_0, w_0). The distance from the field point to the i-th edge is R_i^0; the distance from the field point to the begin and end vertices of the i-th edge is R_i^- and R_i^+ respectively.

The field point can be projected onto the triangle plane. The projection point has a signed distance t_i^0 to the i-th edge, which is positive if the projection point is on the outer side of the edge, while the triangle is located on the interior side of the edge. The distance from the projection point to the begin and end point of the i-th edge is t_i^- and t_i^+ respectively.

Finally, the field point can be projected to the i-th edge. This point defines the origin on the i-th edge where $s_i = 0$. The positions of the begin and end point of the i-th edge are thus defined as s_i^- and s_i^+ respectively.

Graglia [1993] provides the following equations to calculate the R, s and t variables:

$$s_1^- = -\frac{(l_3-u_0)(l_3-u_3)+v_0v_3}{l_1}, \tag{S26}$$

$$s_1^+ = \frac{(u_3-u_0)(u_3-l_3)+v_3(v_3-v_0)}{l_1}, \tag{S27}$$

$$s_2^- = -\frac{u_3(u_3-u_0)+v_3(v_3-v_0)}{l_2}, \tag{S28}$$

$$s_2^+ = \frac{u_0u_3+v_0v_3}{l_2}, \tag{S29}$$

$$s_3^- = -u_0\,,\ s_3^+ = l_3-u_0\,, \tag{S30}$$

$$t_1^0 = \frac{v_0(u_3-l_3)+v_3(l_3-u_0)}{l_1}, \tag{S31}$$

$$t_2^0 = \frac{u_0v_3-v_0u_3}{l_2}, \tag{S32}$$

$$t_3^0 = v_0\,, \tag{S33}$$

$$t_3^+ = t_1^-\,,\ t_2^- = t_1^+\,,\ t_3^- = t_2^+\,, \tag{S34}$$

$$t_1^- = \sqrt{(l_3-u_0)^2+v_0^2}\,, \tag{S35}$$

$$t_t^+ = \sqrt{(u_3-u_0)^2+(v_3-v_0)^2}\,, \tag{S36}$$

$$t_2^+ = \sqrt{u_0^2+v_0^2}\,, \tag{S37}$$

$$R_{[1,2,3]}^0 = \sqrt{w_0^2+(t_{[1,2,3]}^0)^2}\,. \tag{S38}$$

In addition to these basic variables, some auxiliary functions are also introduced:

$$f_{2i} = \ln\frac{R_i^+ + s_i^+}{R_i^- + s_i^-}, \tag{S39}$$

$$\beta_i = \tan^{-1}\frac{t_i^0 s_i^+}{(R_i^0)^2+|w_0|R_i^+} - \tan^{-1}\frac{t_i^0 s_i^-}{(R_i^0)^2+|w_0|R_i^-}, \tag{S40}$$

$$\beta = \beta_1+\beta_2+\beta_3\,. \tag{S41}$$

S2.2 Helmholtz Decomposition

We first discuss the detailed formulas for the full Helmholtz decomposition using analytical integration. Assume the liquid is contained in the domain Ω, whose boundary is Γ, which is already discretized by triangles. Each vertex i is associated with the velocity $\boldsymbol{u}_i$, and a piece-wise linear basis function ψ_i^1 (Figure S1). For such a velocity field defined over the boundary Γ, we can construct the harmonic velocity field defined inside the domain Ω:

$$\phi(\boldsymbol{x}) = -\int_\Gamma \boldsymbol{n}(\boldsymbol{y})\cdot\boldsymbol{u}(\boldsymbol{y})\frac{1}{4\pi}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}ds_{\boldsymbol{y}}\,,\ \boldsymbol{x}\in\Omega\,, \tag{S42}$$

$$\boldsymbol{A}(\boldsymbol{x}) = -\int_\Gamma \boldsymbol{n}(\boldsymbol{y})\times\boldsymbol{u}(\boldsymbol{y})\frac{1}{4\pi}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}ds_{\boldsymbol{y}}\,,\ \boldsymbol{x}\in\Omega\,, \tag{S43}$$

$$\begin{aligned}\tilde{\boldsymbol{u}} &= -\nabla\phi+\nabla\times\boldsymbol{A} \qquad (S44)\\ &= \int_\Gamma \boldsymbol{n}(\boldsymbol{y})\cdot\boldsymbol{u}(\boldsymbol{y})\nabla_{\boldsymbol{x}}\frac{1}{4\pi}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}ds_{\boldsymbol{y}}\\ &\quad -\nabla_{\boldsymbol{x}}\times\int_\Gamma \boldsymbol{n}(\boldsymbol{y})\times\boldsymbol{u}(\boldsymbol{y})\frac{1}{4\pi}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}ds_{\boldsymbol{y}}\\ &= \int_\Gamma \boldsymbol{n}(\boldsymbol{y})\cdot\boldsymbol{u}(\boldsymbol{y})\nabla_{\boldsymbol{x}}\frac{1}{4\pi}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}ds_{\boldsymbol{y}}\\ &\quad +\int_\Gamma(\boldsymbol{n}(\boldsymbol{y})\times\boldsymbol{u}(\boldsymbol{y}))\times\nabla_{\boldsymbol{x}}\frac{1}{4\pi}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}ds_{\boldsymbol{y}}\,. \qquad (S45)\end{aligned}$$

In the discretized form, the boundary Γ consist of triangles, and the velocity field over the boundary are represented by summation of linear bases defined on triangle meshes. Inside the triangle, the

velocity $\boldsymbol{u}$ is linearly interpolated by the velocity values on the three vertices. Meanwhile, the outward normal vector $\boldsymbol{n}$ of a triangle is constant. The integral in Eq. (S45) can be used to evaluate the reconstructed velocity inside the fluid. We need to evaluate the velocity on the fluid boundary. Specifically, we need to evaluate the volume velocity, but take its limit at the interior limit of the fluid boundary, and furthermore integrate it with a test linear basis function defined on the fluid boundary. Hence, in the end, we need to evaluate a double integral over pairs of triangles.

Let us focus on one pair of target triangle T and source triangle T' corresponding to a part of the integral domain of the outer and inner integral respectively. Each triangle has three linear basis functions $\lambda_{[1,2,3]}(\cdot)$ and three vertices have velocities $\boldsymbol{u}_{[1,2,3]}$. The i-th linear basis function $\lambda_i(\cdot)$ has value 1 on the i-th vertex, and value 0 on the i-th edge opposed to the i-th vertex. We focus on the i-th test function λ_i on the target triangle. The ϕ part and the $\boldsymbol{A}$ part of the integral are:

$$
\begin{aligned}
c_j &= \boldsymbol{n}\cdot\boldsymbol{u}_j\,, && \text{(S46)}\\
\boldsymbol{J}_j &= \boldsymbol{n}\times\boldsymbol{u}_j\,, && \text{(S47)}\\
\nabla_{\boldsymbol{x}} &= -\nabla_{\boldsymbol{x}'}\,, && \text{(S48)}
\end{aligned}
$$

$$
\boldsymbol{I}_\phi = -\sum_{j=1}^{3}\frac{1}{4\pi}\int_T \lambda_i \int_{T'} c_j\lambda_j'\nabla'\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}da'da\,, \qquad \text{(S49)}
$$

$$
\boldsymbol{I}_{\boldsymbol{A}} = -\sum_{j=1}^{3}\frac{1}{4\pi}\int_T \lambda_i \int_{T'} \boldsymbol{J}_j\times\lambda_j'\nabla'\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}da'da\,. \qquad \text{(S50)}
$$

There are two scenarios to consider: identical triangles $T = T'$ and different triangles $T \neq T'$. We first consider the different triangle integral. In this case, we use a three point quadrature rule to evaluate the outer integral over triangle T and use analytical integration over T':

$$
\boldsymbol{I}_\phi = -\frac{1}{4\pi}\sum_{k=1}^{3} w_k\lambda_i(\boldsymbol{x}_k)\sum_{j=1}^{3}\int_{T'} c_j\lambda_j'\nabla'\frac{1}{|\boldsymbol{x}_k-\boldsymbol{x}'|}da'\,, \qquad \text{(S51)}
$$

$$
\boldsymbol{I}_{\boldsymbol{A}} = -\frac{1}{4\pi}\sum_{k=1}^{3} w_k\lambda_i(\boldsymbol{x}_k)\sum_{j=1}^{3}\int_{T'} \boldsymbol{J}_j\times\lambda_j'\nabla'\frac{1}{|\boldsymbol{x}_k-\boldsymbol{x}'|}da'\,, \qquad \text{(S52)}
$$

where $\boldsymbol{x}_k$ is the k-th quadrature point on target triangle T, and w_k is the quadrature weight, which is one third of the target triangle area.

They all share the same core, which depends on the source shape function λ_j':

$$
\begin{aligned}
\boldsymbol{I}_{\text{core}}(j) &= \int_{T'}\lambda_j'\nabla'\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}da'\\
&= \boldsymbol{I}_1(j)+\boldsymbol{I}_2(j)\,, && \text{(S53)}\\
\boldsymbol{I}_1(j) &= \boldsymbol{n}\int_{T'}\lambda_j'\frac{w_0}{|\boldsymbol{x}-\boldsymbol{x}'|^3}da'\,, && \text{(S54)}\\
\boldsymbol{I}_2(j) &= \int_{T'}\lambda_j'\nabla_T'\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}da'\,, && \text{(S55)}
\end{aligned}
$$

in which ∇_T' denotes the gradient operator on $\boldsymbol{x}'$ only considering components inside the triangle plane.

The normal component $\boldsymbol{I}_1(j)$ is further decoupled:

$$
\begin{aligned}
\boldsymbol{I}_1(j) &= \boldsymbol{n}(I_3(j)+I_4(j))\,, && \text{(S56)}\\
I_3(j) &= \hat{\boldsymbol{m}}_j\cdot\frac{w_0}{h_j}\sum_{i=1}^{3}\hat{\boldsymbol{m}}_i f_{2i}\,, && \text{(S57)}\\
I_4(j) &= \frac{t_j^0}{h_j}\,\text{sign}(w_0)\beta\,. && \text{(S58)}
\end{aligned}
$$

The tangential component $\boldsymbol{I}_2(j)$ can be decoupled as follows:

$$
\begin{aligned}
\boldsymbol{I}_2(j) &= \boldsymbol{I}_5(j)+\boldsymbol{I}_6(j)\,, && \text{(S59)}\\
\boldsymbol{I}_5(j) &= \int_{T'}\nabla_T'\left(\frac{\lambda_j'}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)da'\,, && \text{(S60)}\\
\boldsymbol{I}_6(j) &= \frac{\hat{\boldsymbol{m}}_j}{h_j}\int_{T'}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}da'\,. && \text{(S61)}
\end{aligned}
$$

The expanded form of $\boldsymbol{I}_5(j)$ is given by

$$
\begin{aligned}
\int_{T'}\nabla_T'\left(\frac{\lambda_j'}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)da' &= \sum_{i=1}^{3}\hat{\boldsymbol{m}}_i\int_{\partial T'}\frac{\lambda_j'}{|\boldsymbol{x}-\boldsymbol{x}'|}dl'\\
&= (R_{j+1}^{+}-R_{j+1}^{-}-s_{j+1}^{-}f_{2j+1})\frac{\hat{\boldsymbol{m}}_{j+1}}{l_{j+1}}\\
&+ (R_{j-1}^{-}-R_{j-1}^{+}+s_{j-1}^{+}f_{2j-1})\frac{\hat{\boldsymbol{m}}_{j-1}}{l_{j-1}}\,. \qquad \text{(S62)}
\end{aligned}
$$

Finally, $\boldsymbol{I}_6(j)$ contains a single layer potential with constant charges over the triangle:

$$
\boldsymbol{I}_6(j) = \frac{\hat{\boldsymbol{m}}_j}{h_j}(-|w_0|\beta+\sum_{i=1}^{3}t_i^0 f_{2i})\,. \qquad \text{(S63)}
$$

$\boldsymbol{I}_{\text{core}}(j)$ has to be multiplied with c_j and cross-producted by $\boldsymbol{J}_j$ to conclude the computation in Eq. (S51) and (S52).

In another scenario, the source and target triangles overlap. Furthermore the field point should be the interior limit of the triangle; in other words: $w_0 \to 0^-$. Please note, that $\boldsymbol{I}_{\text{core}}$ is part of the inner integral. For the outer integral, terms can benefit from a more analytical integral expressions. For example, $I_3 = 0$ because $w_0 = 0$. For the I_4 term, inside the triangle, $\beta = 2\pi$, while $\text{sign}(w_0) = -1$ if $\boldsymbol{x}$ is approaching from the interior side of the triangle. We have not yet found a simplification of $\int_T \lambda_i \boldsymbol{I}_5(j)da$, so that the outer integral is still evaluated using quadrature rules. For $\boldsymbol{I}_6$, the double integral reads:

$$
\frac{\hat{\boldsymbol{m}}_j}{h_j}\int_T\lambda_i\int_{T'}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}da'da\,. \qquad \text{(S64)}
$$

The double integral is provided in Sievers et al. [2005] and Eq. (8) therein.

S2.3 Magnetic and Pressure Problem

When evaluating the operator K in the context of the magnetic problem, the integration of a double layer potential of linearly varying charges integrated over a triangle is required. The related formula is the scalar part of the $\boldsymbol{I}_1$ integral in Eq. (S54).

To evaluate the gradient of the double layer potential with a charge distribution given by a linear basis function ψ^1, the surface curl of a scalar function over a triangle is required, and the integral

of the gradient of a single layer potential with constant charges over a triangle. Assume the triangle has three charges c_1, c_2, c_3 on its three vertices, $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3$. The charges are linearly interpolated on the triangle so the gradient of the scalar field is constant on the triangle. The surface curl is given by

$$\mathrm{curl}_T(c) = \frac{1}{2\,\mathrm{Area}}((c_3 - c_2)\boldsymbol{r}_1 + (c_1 - c_3)\boldsymbol{r}_2 + (c_2 - c_1)\boldsymbol{r}_3)\,. \quad \text{(S65)}$$

The gradient of single layer potential with constant charges can be written in the following form:

$$\begin{aligned} & \int_{T'} \nabla \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} da' \\ = & -\int_{T'} \nabla' \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} da' \\ = & -(\boldsymbol{n} w_0 \int_{T'} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|^3} da' + \nabla'_{T'} \int_{T'} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} da') \\ = & -(\boldsymbol{n}\,\mathrm{sign}(w_0)\beta + \sum_{i=1}^{3} \hat{\boldsymbol{m}}_i f_{2i})\,. \end{aligned} \quad \text{(S66)}$$

When evaluating the gradient at the quadrature point inside the source triangle, $\mathrm{sign}(w_0) = -1$ and $\beta = 2\pi$.

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