Supplementary: Ships, Splashes, and Waves on a Vast Ocean

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S1 JUSTIFICATION OF BEM FORMULATIONS

In this section we use the acronym BEM for the surface only liquid solver, as well as literally for boundary element methods. Here we present a theory explaining why the implementation from Da et al. [2016] is used instead of the one from Huang and Michels [2020] for our purpose, though the latter implementation actually satisfies the original assumption of Da et al. [2016], while the original implementation does not.

The governing equation of the BEM is the incompressible Euler equation [Bridson and Müller-Fischer 2007]:

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho}\nabla P + \boldsymbol{g}\,, \tag{S1}$$

$$\nabla \cdot \boldsymbol{u} = 0\,, \tag{S2}$$

where $D/Dt := \partial/\partial t + \boldsymbol{u}\cdot\nabla$ is the material derivative, $\boldsymbol{u}$ is the velocity field, ρ is the constant liquid density, P is the pressure, and g is the gravitational acceleration.

In the BEM, the equation is solved by operator splitting in an advection-projection style. Assume that in the beginning of the time step, the velocity is $\boldsymbol{u}^- = \boldsymbol{u}(t)$. After the advection, the velocity becomes $\boldsymbol{u}$. Finally, after the projection, the velocity becomes $\boldsymbol{u}^+ = \boldsymbol{u}(t+\Delta t)$, where Δt is the time step size.

In the advection step, we solve the advection equation

$$\frac{D\boldsymbol{u}}{Dt} = 0 \tag{S3}$$

to obtain $\boldsymbol{u}$ from $\boldsymbol{u}^-$, and then add gravity to it. In the pressure projection step, the incompressibility constraint $\nabla\cdot\boldsymbol{u} = 0$ is enforced by solving $\partial\boldsymbol{u}/\partial t = -\rho^{-1}\nabla P$. The Poisson's equation is

$$\begin{aligned} \nabla^2 P &= \frac{\rho}{\Delta t}\nabla\cdot\boldsymbol{u}\,, \\ P|_{\Gamma_\mathrm{D}} &= P_\mathrm{bc}\,, \\ \frac{dP}{dn}|_{\Gamma_\mathrm{N}} &= \frac{\rho}{\Delta t}(\boldsymbol{u}-\boldsymbol{u}_\mathrm{solid})\cdot\boldsymbol{n}\,, \end{aligned} \tag{S4}$$

where Γ_D and Γ_N are liquid-air (Dirichlet) and liquid-solid (Neumann) boundaries respectively, P_bc is the Dirichlet pressure boundary condition which usually consists of surface tension, $\boldsymbol{u}_\mathrm{solid}$ is the desired liquid velocity on the solid boundary, which can introduce inflow and outflow, and $\boldsymbol{n}$ is the external normal vector of the liquid domain. Eq. (S4) is solved to get the pressure P, and the velocity is then updated by

$$\boldsymbol{u}(t+\Delta t) = \boldsymbol{u}^+ = \boldsymbol{u} - \frac{\Delta t}{\rho}\nabla P\,. \tag{S5}$$

In the BEM, the liquid is represented by triangle meshes delineating its boundary. The velocity of the boundary is stored on each vertex. The core idea of this method is to evolve the boundary mesh with the vertex velocity, and only update the velocity of the vertices on the liquid boundary. In the advection step, each vertex moves to its new position according to its velocity.

The pressure projection Eq. (S4) is effectively solved in two steps. Eq. (S4) is equivalent to the following two systems if we define $P := P_1 + P_2$. Homogeneous Dirichlet boundary conditions apply for P_1 with non-zero right hand side, while P_2 has zero right hand side and mixed boundary conditions. The first system is given by

$$\begin{aligned} \nabla^2 P_1 &= \frac{\rho}{\Delta t}\nabla\cdot\boldsymbol{u}\,, \\ P_1|_{\Gamma_\mathrm{D}} &= 0\,, \end{aligned} \tag{S6}$$

and the second system by

$$\begin{aligned} \nabla^2 P_2 &= 0\,, \\ P_2|_{\Gamma_\mathrm{D}} &= P_\mathrm{bc}\,, \\ \frac{dP_2}{dn}|_{\Gamma_\mathrm{N}} &= \frac{\rho}{\Delta t}(\boldsymbol{u}-\boldsymbol{u}_\mathrm{solid})\cdot\boldsymbol{n} - (\nabla P_1)\cdot\boldsymbol{n} \\ &= \frac{\rho}{\Delta t}((\boldsymbol{u} - \frac{\Delta t}{\rho}\nabla P_1) - \boldsymbol{u}_\mathrm{solid})\cdot\boldsymbol{n} \\ &= \frac{\rho}{\Delta t}(\tilde{\boldsymbol{u}} - \boldsymbol{u}_\mathrm{solid})\cdot\boldsymbol{n}\,. \end{aligned} \tag{S7}$$

The first system corrects the divergence in the right hand side of Eq. (S6) with zero Dirichlet boundary conditions. After we solve the first system, we obtain an intermediate velocity

$$\tilde{\boldsymbol{u}} = \boldsymbol{u} - \frac{\Delta t}{\rho}\nabla P_1\,. \tag{S8}$$

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The intermediate velocity $\tilde{\boldsymbol{u}}$ is then plugged into the second system, which is a Laplace equation with mixed boundary conditions.

The surface-only liquids method of Da et al. [2016] applies two steps within the pressure projection procedure corresponding to the above two systems. The first step is called a Helmholtz decomposition (H.D.) which projects the velocity field $\boldsymbol{u}$ into a divergence-free and curl-free velocity field. Next, the boundary element method is used to solve Eq. (S7) for P_2 because it is one of its model problems. After P_2 is solved, the gradient is used to update the intermediate velocity to obtain the final velocity after pressure projection: $\boldsymbol{u}^+ = \tilde{\boldsymbol{u}} - \Delta t/\rho\, \nabla P_2$.

The remaining task is to find the connection between the H.D. and the first set of pressure Eq. (S6). We will see that the H.D. from Da et al. [2016] approximately solves Eq. (S6) for P_1.

With the help of Green's representation theorem, the solution P_1 can be written as

$$P_1 = \int_\Omega -\frac{\rho}{\Delta t} \frac{\nabla_y \cdot \boldsymbol{u}(y)}{4\pi|x-y|} dy + \int_\Gamma \frac{\partial P_1(y)/\partial n}{4\pi|x-y|} ds_y\,, \tag{S9}$$

where Ω is the liquid volume and $\Gamma = \partial\Omega$ is the liquid boundary. The first term is the volume integral of divergence multiplied with Green's function $1/(4\pi|x-y|)$. The important message of this equation is that the true solution P_1 without any approximation consists of a volume divergence integral as the first part and a second part with unknown normal derivative $\partial P_1/\partial n$ which ensures on the boundary Γ, $P_1 = 0$, so that the gradient only has a normal component. Its gradient ∇P_1 is used to apply the velocity update Eq. (S8), which removes all the volume divergence, and has only normal components on the boundary Γ.

The ∇P_1 update shares some similarity with the update of the H.D. To see that we take a look at the definition of the H.D. for the velocity field $\boldsymbol{u}$. The following equation holds for general $\boldsymbol{u}$:

$$\begin{aligned}\boldsymbol{u}(x) \;=\; & -\nabla \left(\int_\Omega \frac{\nabla_y \cdot \boldsymbol{u}(y)}{4\pi|x-y|} dy - \int_\Gamma \frac{\boldsymbol{n}(y) \cdot \boldsymbol{u}(y)}{4\pi|x-y|} ds_y \right) \\ & + \nabla \times \left(\int_\Omega \frac{\nabla_y \times \boldsymbol{u}(y)}{4\pi|x-y|} dy - \int_\Gamma \frac{\boldsymbol{n}(y) \times \boldsymbol{u}(y)}{4\pi|x-y|} ds_y \right) . \end{aligned} \tag{S10}$$

Da et al. [2016] assume the liquid to be divergence-free and curl-free. Under this assumption, the volume divergence and curl integral in Eq. (S10) vanish, and the velocity can be uniquely determined by the velocity on the boundary. The outcome of this assumption is the velocity

$$\boldsymbol{u}_{\text{HD}}(x) = \nabla \int_\Gamma \frac{\boldsymbol{n}(y) \cdot \boldsymbol{u}(y)}{4\pi|x-y|} ds_y - \nabla \times \int_\Gamma \frac{\boldsymbol{n}(y) \times \boldsymbol{u}(y)}{4\pi|x-y|} ds_y\,, \tag{S11}$$

which implies a correction $\delta \boldsymbol{u}_{\text{hd}}$ that removes both the divergence and curl in the liquid volume:

$$\delta\boldsymbol{u}_{\text{HD}}(x) = \nabla \int_\Omega \frac{\nabla_y \cdot \boldsymbol{u}(y)}{4\pi|x-y|} dy - \nabla \times \int_\Omega \frac{\nabla_y \times \boldsymbol{u}(y)}{4\pi|x-y|} dy\,. \tag{S12}$$

While Huang and Michels [2020] stick to the divergence-free and curl-free assumption, and apply the correction Eq. (S12) as is, Da et al. proposed to only keep the correction in the surface normal direction. We believe that for our purpose the method of Da et al. is more suitable, because the ideal update $-\Delta t/\rho\, \nabla P_1$ only contains corrections in the normal direction. These two approaches are referred to as full and partial H.D. respectively.

The correction in the full H.D. in Eq. (S12) and the ideal pressure gradient correction based on Eqs. (S8-S9) share the same volume divergence integral as the first part. On the liquid boundary, the gradient of the divergence integral contains both tangential and normal components. However, the ideal pressure Eq. (S9) has a second boundary integral. The gradient of the second boundary integral is divergence-free, curl-free, and cancels the tangential component of the volume divergence integral on the boundary. Therefore, taking the correction of the full H.D. in Eq. (S12) and only keep the normal component on the boundary is an approximation to the ideal pressure correction in Eq. (S9).

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